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Introduction and Proof of Taylor's Theorum

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Taylor's Theorem is a very powerful tool to approximate any functions that are infinitely differentiable on a certain interval between a and b. Of course, the exact value of a and b need to be carefully defined, so the formula/series developed by the theorem shall converge within the defined interval.

1 Description

Taylor's Theorem If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between$a$ and b, then there exists a number c between a and b such that

$$\begin{aligned} f(b) = & f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots \\ & + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \end{aligned}$$

A common formula for Taylor's Theorem usually use x instead of b, and $R_n(x)$ to stand for the remainder term:

$$\begin{aligned} f(x) = & f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ & + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \end{aligned}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x.$$

Furthermore, if we let $a = 0$, the above Taylor's formula reduces to **Maclaurin series** (a special case of Taylor's series):

$$f(x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \cdots + \frac{f^{(n)}(a)}{n!}x^n + R_n(x)$$

2 An intuitive Explanation and Error Estimate (Convergence)

At point a, if Taylor's polynomial formula and its derivatives (infinitely) have the same value as the original function, maybe the two functions will perform quite the same around point a. I think this is the logic behind the theorem, and now let's take a look at the remainder $R_n(x)$ and see if and under what conditions it goes to zero as $n \to \infty$, which means the Taylor's formula would be a "perfect" approximation for the original function.

Lemma If there is a positive constant M such that $|f^{(n+1)(c)}| \leqslant M$ for all c between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality and goes to zero as $n \to \infty$:

$$|R_n(x)| \leqslant M \frac{|x-a|^{n+1}}{(n+1)!}$$

The above conclusion directly derives from the convergence of the form $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$. So it's also true that if we choose x carefully so that $|f^{(n+1)(c)}|$ is equal to or less than a constant, the Taylor's formula holds for the function.

3 Applications of Taylor's Formula

Use Taylor's formula, especially when we pick $a = 0$, will give us some every useful series.

Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \quad , \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-x)^n \quad , \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad , \quad |x| < \infty$$

$$\cos x = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad , \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad , \quad -1 < x \leqslant 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \quad |x| \leqslant 1$$

4 Proof of Taylor's Theorem

In order to approximate function f, let a polynomial function P be:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$P_n(x)$ matches function f for the first n derivatives at $x = a$, and if we construct another function $\phi(x)$ by adding the $(n+1)$th term, the matching still holds

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}$$

Matching of functions $f(x)$ and $\phi_n(x)$ at $x = a$ doesn't lead to the conclusion that they also math at $x = b$. However, we can pick a value for K to let the equation hold,

$$f(b) = \phi_n(b) = P_n(b) + K(b-a)^{n+1} \qquad \text{when} \qquad K = \tfrac{f(b)-P_n(b)}{(b-a)^{n+1}}$$

As long as a and b are fixed, we know K would be a constant. Now we need to construct a third function $G(x)$ to find this constant K to a more precise extent:

$$G(x) = f(x) - \phi_n(x) = f(x) - P_n(x) - K(x-a)^{n+1} \tag{A}$$

$G(x)$ actually measures the difference between the original function f and the Taylor's series formula approximation, and it has two features by its nature:

(1) $G(a) = f(a) - \phi_n(a) = 0$; and $G(b) = f(b) - \phi_n(b) = 0$;

(2) $G'(a) = G''(a) = \cdots = G^{(n)}(a) = 0$

From (1), we know that by Rolle's Theorem, there must be a point c_1 between a and b that $G'(c_1) = 0$, and from (2) we can conclude that there must be a

$$\begin{gathered} c_2 \text{ in } (a, c_1) \text{ such that } G^{(2)}(c_2) = 0, \\ c_3 \text{ in } (a, c_2) \text{ such that } G^{(3)}(c_3) = 0, \\ \vdots \\ c_{n+1} \text{ in } (a, c_n) \text{ such that } G^{(n+1)}(c_{n+1}) = 0 \end{gathered}$$

We know that the $(n+1)$th derivative of $\phi_n(x)$ is

$$\phi_n^{(n+1)}(c_{n+1}) = P_n^{(n+1)}(x) + K \cdot (n+1)! = K \cdot (n+1)!$$

and by Eq. (A) we have

$$G^{(n+1)}(c_{n+1}) = 0 = f^{(n+1)}(c_{n+1}) - K \cdot (n+1)!$$

$$K = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}$$