The Negative Binomial Dsitribution

Study Notes | Written by Larry Cui

Follow the logic of geometric distribution, if we want to study the probability of rth success in k^{th} of a series of trials, it must be the case that $(r-1)$ success occur during the first $(k-1)$ trials and the rth happens on exactly the kth trial.

Negative Binomial Distribution pdf

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \qquad k = r, r+1, \ldots$$

If we let X be the sum of independent variables $X_1, X_2, \ldots, X_r$, and let $X \to \infty$, the negative binomial can be interpreted as r successes happen one after another, and each of which follows the geometric distribution model. This perspective won't give us a better form to the pdf, but will greatly simplify the calculation of expected values.

E(X), Var(X) and mgf

Let $p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$ then:

a. $M_X(t) = M_{X_1}(t) M_{X_2}(t) \ldots M_{X_r}(t) = \left[\dfrac{pe^t}{1 - (1-p)e^t} \right]^r$

b. $E(X) = E(X_1) + E(X_2) + \cdots + E(X_r) = \dfrac{r}{p}$

c. $\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_r) = \dfrac{r(1-p)}{p^2}$

Algebraic Approach: the sum of pdf

There's an algebraic approach to the $E(X)$ and $\mathrm{Var}(X)$, which centres around the verification of the pdf of the function. Let's take a look at the following lemma:

lemma (a)

$$\binom{k+m}{m} = \binom{k+m-1}{m} + \binom{k+m-1}{m-1}$$

Proof: a simple expansion of the combinations gives us result. Furthermore, we can also see this equation by some intuition: if you want pick m items from $k+m$ pool, you can first

split the pool into one item and the other. The final pick consists of two scenarios: 1) excluding this one item: $\begin{pmatrix} k+m-1 \\ m \end{pmatrix}$; and 2) including this one item: $\begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix}$.

$$\begin{pmatrix} k+m-1 \\ m \end{pmatrix} + \begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix} = \frac{(k+m-1)!}{m!(k-1)!} + \frac{(k+m-1)!}{(m-1)!k!}$$
$$= \frac{(k+m-1)!k + (k+m-1)!m}{m!k!}$$
$$= \frac{(k+m)!}{m!k!}$$

Intermediate function: consider the function

$$f_m(z) = \sum_{k=0}^{\infty} \begin{pmatrix} k+m \\ m \end{pmatrix} z^k$$

Use lemma (a), we can expand it as

$$f_m(z) = \sum_{k=0}^{\infty} \begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix} z^k + \sum_{k=0}^{\infty} \begin{pmatrix} k+m-1 \\ m \end{pmatrix} z^k$$

Because $\begin{pmatrix} m-1 \\ m \end{pmatrix} = \frac{(m-1)(m-2)\cdots(m-m)}{m!(-1)!} = 0$ by definition, we can discard the $k=0$ item from the second term. A little re-arrangement give the following form for the second term:

$$\sum_{k=1}^{\infty} \begin{pmatrix} k+m-1 \\ m \end{pmatrix} z^k = z \sum_{k=1}^{\infty} \begin{pmatrix} k+m-1 \\ m \end{pmatrix} z^{k-1} = z f_m(z)$$

so

$$f_m(z) = f_{m-1}(z) + z f_m(z) \quad \Rightarrow \quad f_m(z) = \frac{f_{m-1}(z)}{1-z}$$

But since $f_0(z) = \sum_{k=0}^{\infty} \begin{pmatrix} k \\ 0 \end{pmatrix} z^k = \frac{1}{1-z}$, we know $f_m(z) = \frac{1}{(1-z)^{m+1}}$.

To verify the sum of the pdf equals to 1, we use the above function to get (here we let $m = r-1, k' = k-r$):

$$\sum_{k=r}^{\infty} \begin{pmatrix} k-1 \\ r-1 \end{pmatrix} p^r (1-p)^{k-r} = p^r \cdot \sum_{k'=0}^{\infty} \begin{pmatrix} k'+m \\ m \end{pmatrix} (1-p)^{k'} \tag{1}$$
$$= p^r \cdot f_m(1-p)$$
$$= p^r \cdot \frac{1}{(1-1+p)^r} = 1 \qquad \text{Remark: } r = m+1$$

Now we can go on with $E(X)$:

$$E(X) = \sum_{k=r}^{\infty} \binom{k-1}{r-1} k p^r (1-p)^{k-r} \tag{2}$$

$$= p^r \cdot \sum_{k'=0}^{\infty} (k'+r) \binom{k'+m}{m} (1-p)^{k'}$$

$$= p^r \cdot \sum_{k'=0}^{\infty} r \binom{k'+r}{r} (1-p)^{k'} \qquad \text{Remark: } m = r-1$$

$$= rp^r \cdot \frac{1}{(1-1+p)^{r+1}} = \frac{r}{p}$$

In order to find $\text{Var}(X)$, we need a little trick here: find $E[X(X+1)]$ first. As above, we start by lay out the formula:

$$E[X(X+1)] = \sum_{k=r}^{\infty} k(k+1) \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$= p^r \cdot \sum_{k'=0}^{\infty} (k'+r+1)(k'+r) \binom{k'+m}{m} (1-p)^{k'} \qquad \text{Remark: } k = k'+r$$

$$= p^r \cdot \sum_{k'=0}^{\infty} (k'+r+1)(k'+r) \binom{k'+r-1}{r-1} (1-p)^{k'} \quad \text{Remark: } m = r-1$$

$$= r(r+1)p^r \cdot \sum_{k'=0}^{\infty} \binom{k'+r+1}{r+1} (1-p)^{k'}$$

$$= r(r+1)p^r \cdot \frac{1}{(1-1+p)^{r+2}}$$

$$= \frac{r(r+1)}{p^2}$$

so we have

$$\text{Var}(X) = E[X(X+1)] - E(X)^2 - E(X) = \frac{r(r+1)}{p^2} - \frac{r^2}{p^2} - \frac{r}{p} = \frac{r(1-p)}{p^2}$$

A second way:

We can also find the solution to Equation (1) without having recourse to intermediate function $f_m(z)$. First of all, we re-arrange the expected value formula as:

$$E(X) = \sum_{k=r}^{\infty} \binom{k-1}{r-1} k p^r (1-p)^{k-r}$$

$$= p^r \cdot \sum_{k'=0}^{\infty} (k'+r) \binom{k'+r-1}{r-1} (1-p)^{k'} \qquad \text{Remark: } k' = k-r$$

$$= rp^r \cdot \sum_{k'=0}^{\infty} \binom{k'+(r+1)-1}{k'} (1-p)^{k'}$$

$$= rp^r \cdot \sum_{k'=0}^{\infty} \binom{k'+(r+1)-1}{k'} (-1)^{k'} (p-1)^{k'}$$

We need Taylor/Maclaurin series for help here. Recall

The Maclaurin series for $f(x)$

wherever it converges, can be expressed as:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

and for $f(x) = 1/(1+x)^n$:

$$\frac{1}{(1+x)^n} = 1 - nx + \frac{(-n)(-n-1)}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \binom{n+k-1}{k}(-1)^k x^k$$

We can see that the last part of $E(X)$ is exactly the sum of Maclaurin series with $n = r+1$ and $x = p - 1$, so

$$E(X) = rp^r \cdot \frac{1}{(1+p-1)^{r+1}} = \frac{r}{p}$$