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PDF of Sum, Quotient and Difference of Multiple Variables

Study Notes | Larry Cui

1 Transformation

theorem

Suppose X is a *continuous random variable.* Let $Y = aX + b$. Then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Proof:

We begin by writing an expression for the cdf of Y:

$$F_Y(y) = P(Y \leqslant y) = P(aX + b \leqslant y) = P\left(X \leqslant \frac{y-b}{a}\right) \qquad \triangleright\ a > 0$$

Differentiating $F_Y(y)$ yields $f_Y(y)$:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) = \frac{dx}{dy}f_X\left(\frac{y-b}{a}\right) = \frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

However, when $a < 0$, $F_Y(y) = P\left(X > \frac{y-b}{a}\right) = 1 - P\left(X \leqslant \frac{y-b}{a}\right)$. Differentiating it the same way we can get:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right) = -\frac{dx}{dy}f_X\left(\frac{y-b}{a}\right) = -\frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

Combine the two situations: $f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$.

2 PDF of a Sum

theorem

Suppose that X and Y are *independent random variables.* Let $W = X + Y$. Then

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx$$

Proof:

We start with cdf again:

$$\begin{aligned} F_W(w) &= \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_X(x) f_Y(y)\, dy\, dx \\ &= \int_{-\infty}^{\infty} f_X(x) \left[\int_{-\infty}^{w-x} f_Y(y)\, dy \right] dx \\ &= \int_{-\infty}^{\infty} f_X(x) F_Y(w-x)\, dx \end{aligned}$$

Differentiating $F_W(w)$ to get $f_W(w)$:

$$\begin{aligned} f_W(w) = \frac{d}{dw} F_W(w) &= \frac{d}{dw} \int_{-\infty}^{\infty} f_X(x) F_Y(w-x)\, dx \\ &= \int_{-\infty}^{\infty} f_X(x) \left[\frac{d}{dw} F_Y(w-x) \right] dx \\ &= \int_{-\infty}^{\infty} f_X(x) \left[\frac{dy}{dw} f_Y(w-x) \right] dx \\ &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx \qquad \triangleright\ y = w - x \rightarrow \frac{dy}{dw} = 1 \end{aligned}$$

3 PDF of Quotient

theorem

Suppose that X and Y are *independent random variables*. Let $W = Y/X$. Then

$$f_W(w) = |x| \int_{-\infty}^{\infty} f_X(x) f_Y(wx)\, dx$$

Proof:

We start with cdf. As we know, $F_W(w) = P(Y/X \leqslant w)$. We need to calculate the cdf for $X \geqslant 0$ and $x < 0$ separately, and combine the results of two situations together.

(1)$X \geqslant 0$:
$P(Y/X \leqslant w) = P(Y \leqslant wX) = \int_0^{\infty} \int_{-\infty}^{wx} f_X(x) f_Y(y)\, dy\, dx$

(2)$X < 0$:
$P(Y/X \leqslant w) = P(Y > wX) = 1 - P(Y \leqslant wX) = 1 - \int_{-\infty}^{0} \int_{-\infty}^{wx} f_X(x) f_Y(y)\, dy\, dx$

Now we can have the cdf of $F_W(w)$:

$$F_W(w) = \int_0^{\infty} \int_{-\infty}^{wx} f_X(x) f_Y(y)\, dy\, dx + 1 - \int_{-\infty}^{0} \int_{-\infty}^{wx} f_X(x) f_Y(y)\, dy\, dx$$

Then we differentiate $F_W(w)$ to obtain:

$$
\begin{aligned}
f_W(w) = \frac{d}{dw}F_W(w) &= \frac{d}{dw}\int_0^{\infty}\int_{-\infty}^{wx} f_X(x)f_Y(y)\,dy\,dx - \frac{d}{dw}\int_{-\infty}^{0}\int_{-\infty}^{wx} f_X(x)f_Y(y)\,dy\,dx \\
&= \int_0^{\infty} f_X(x)\left(\frac{d}{dw}\int_{-\infty}^{wx} f_Y(y)\,dy\right)dx - \int_{-\infty}^{0} f_X(x)\left(\frac{d}{dw}\int_{-\infty}^{wx} f_Y(y)\,dy\right)dx \\
&= \int_0^{\infty} f_X(x)\left(\frac{dy}{dw}f_Y(wx)\right)dx - \int_{-\infty}^{0} f_X(x)\left(\frac{dy}{dw}f_Y(wx)\right)dx \\
&= \int_0^{\infty} f_X(x)\left(xf_Y(wx)\right)dx - \int_{-\infty}^{0} f_X(x)\left(xf_Y(wx)\right)dx \qquad \triangleright\ y = wx \to \frac{dy}{dw} = x \\
&= \int_0^{\infty} |x|f_X(x)f_Y(wx)\,dx + \int_{-\infty}^{0} |x|f_X(x)f_Y(wx)\,dx \\
&= \int_{-\infty}^{\infty} |x|f_X(x)f_Y(wx)\,dx
\end{aligned}
$$

4 PDF of Product

theorem

Suppose that X and Y are *independent random variables*. Let $W = XY$. Then

$$f_W(w) = \frac{1}{|x|}\int_{-\infty}^{\infty} f_X(x)f_Y(w/x)\,dx$$

Proof:

We start with cdf. As we know, $F_W(w) = P(XY \leqslant w)$. We also need to calculate the cdf for $X \geqslant 0$ and $x < 0$ separately, and combine the results of two situations together.

(1)$X \geqslant 0$:

$P(XY \leqslant w) = P(Y \leqslant w/X) = \int_0^{\infty}\int_{-\infty}^{w/x} f_X(x)f_Y(y)\,dy\,dx$

(2)$X < 0$:

$P(XY \leqslant w) = P(Y > w/X) = 1 - P(Y \leqslant w/X) = 1 - \int_{-\infty}^{0}\int_{-\infty}^{w/x} f_X(x)f_Y(y)\,dy\,dx$

Now we can have the cdf of $F_W(w)$:

$$F_W(w) = \int_0^{\infty}\int_{-\infty}^{w/x} f_X(x)f_Y(y)\,dy\,dx + 1 - \int_{-\infty}^{0}\int_{-\infty}^{w/x} f_X(x)f_Y(y)\,dy\,dx$$

Then we differentiate $F_W(w)$ to obtain:

$$
\begin{aligned}
f_W(w) = \frac{d}{dw}F_W(w) &= \frac{d}{dw}\int_0^{\infty}\int_{-\infty}^{w/x} f_X(x)f_Y(y)\,dy\,dx - \frac{d}{dw}\int_{-\infty}^{0}\int_{-\infty}^{w/x} f_X(x)f_Y(y)\,dy\,dx \\
&= \int_0^{\infty} f_X(x)\left(\frac{1}{x}f_Y(w/x)\right)dx - \int_{-\infty}^{0} f_X(x)\left(\frac{1}{x}f_Y(w/x)\right)dx \qquad \triangleright\ y = w/x \to \frac{dy}{dw} = \frac{1}{x} \\
&= \int_{-\infty}^{\infty} \frac{1}{|x|}f_X(x)f_Y(w/x)\,dx
\end{aligned}
$$